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A BOUNDARY-INTEGRAL EQUATION APPROACH FOR FOUNDATIONS RESTING ON A DEFORMABLE HALF-SPACE WITH LIMIT CONTACT PRESSURE

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SUMMARY

A spatial contact model for an elastic base which takes into account the limit contact pressure in soil is proposed. Approximate equations permitting the use of the contact model for the description of non-linear 'load-settlement' dependence are presented. The application of the proposed model for calculating contact pressures, settlements and slopes of rigid punches of an arbitrary shape in a plan with the use of boundary-integral equation method leads to systems of non-linear algebraic equations of a special form. Iterative methods of solutions and convergence behaviour of iterations are considered. The approach developed is illustrated by the numerical solution of the contact problem for a circular punch on a non-linearly deformable half-space and a layer of finite thickness. Graphs of contact pressures and the dependence of punch settlement on a vertical load for various values of model parameters are given. Conclusion concerning identification of model parameters for various soil bases are drawn on the basis of punch tests.

KEYWORDS: contact model; boundary-integral equations; foundations

1. INTRODUCTION

For rational design of bases and foundations it is important to take into account the physical non-linearity of soils.[1,2] However, at present there is no unique and reliable approach to the solution of spatial problems of soil mechanics taking into account a wide variety of strength and deformation properties. As a rule, it is connected with the absence of reliable data of soil mechanical characteristics for complicated stress–strain states. In addition, it is also connected with the high level of computer resource consumption. Moreover, one should mention the limited usefulness of the available software which require a large volume of information in input and output at three-dimensional finite element simulation. Therefore the elaboration of more simple spatial contact models, which are able to consider non-linear properties of real bases is quite urgent. The use of such models, together with the effective boundary-integral equation method which render it possible to reduce spatial dimensionality of the problem, will allow the considerable simplification of the calculation method for a number of practical situations. It will also allow a reduction in the volume of computations and a rather exact estimation of the qualitative and quantitative aspects of phenomena of contact constructions interaction with the soil massif. Besides, as it will be shown later, this approach widely uses traditional numerical methods which

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are extensively practised for solving linear problems for a large number of contact models taking into account distributive properties of foundations.

2. SPATIAL CONTACT MODEL FOR NON-LINEARLY DEFORMABLE BASE

In estimating soil foundations in the linear stage of deformation, spatial contact models for half-space,[3,4] layer of constant and variable thickness,[5,6] layered and heterogeneous half-spaces,[7,8] etc., are widely used. The noted contact models represent the solutions of problems concerning the base surface settlement due to the action of concentrated (vertical and horizontal) loads. Contact models, obtained on the basis of strict solutions of mixed spatial problems of elasticity theory, are a rather simple and convenient way to represent the relationship between the pressure p on the base and its settlement W in the form

$$W(x, y) = \frac{1-\nu^2}{\pi E} \int\int_S p(\xi, \eta)\omega(x, y, \xi, \eta) \, d\xi \, d\eta \tag{1}$$

where the function $\omega(x, y, \xi, \eta)$ is a Green's function chosen versus the adopted base model. The contact models employed are linear, and they are distinguished in prediction of the distributive base properties. For real conditions of soil deformation with increasing load, there appears a non-linear character in the 'load-settlement' relation and representation (1) requires generalization.

We will assume that the settlement at a given point of the base surface depends on the level of loading p/p_* and may be presented as follows:

$$d\,W(x, y) = \varphi(p, p_*)\omega(x, y, \xi, \eta) \, d\xi \, d\eta$$

where $\varphi(p, p_*)$ is a loading level function which determines the law of non-linear base surface deformation under a load; p_* is a limit contact stress on the base surface; ξ and η are co-ordinates of the centre of elementary area; x and y are co-ordinates of the point being considered. From mechanical considerations, we can draw qualitative conclusions about the form and properties of $\varphi(p, p_*)$. Thus,

$$\varphi(p, p_*) \sim p, \quad \text{for } p \ll p_*$$

$$\varphi(p, p_*) \to \infty, \quad \text{for } p \to p_*$$

$$\varphi'_p(p, p_*) > 0, \quad \varphi''_{pp}(p, p_*) > 0, \quad 0 < p < p_*$$

We named the value p_* as a limit contact pressure because it is considered as a basic parameter of the contact model. It has dimension of stress and plays the role of a characteristic scale.

The function $\varphi(p, p_*)$ can be described by the following analytical approximation:

$$\varphi(p, p_*) = p/[1 - (p/p_*)^n]^m \tag{2}$$

where m, $n > 0$ are the dimensionless parameters which, together with p_*, are subjected to identification on the basis of experimental data. Later on for definiteness sake, we will assume $n = 1$. Figure 1 shows dimensionless parametric dependencies $\varphi(p, p_*)$, reflecting the character of their non-linear behaviour as the parameter m increases. It should be noted that at small values of m we have the case of a linear 'load-settlement' relation practically throughout the entire range of acting stresses.

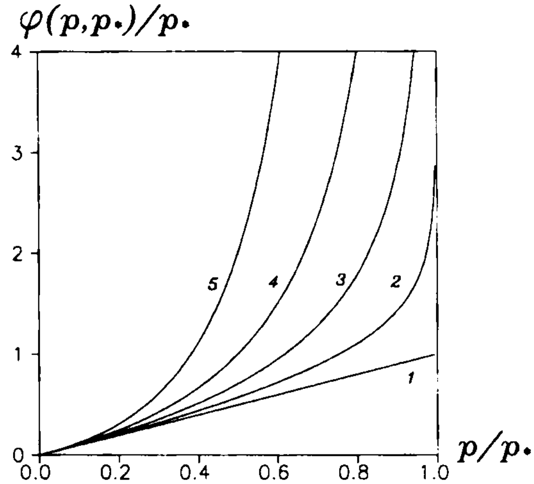

Figure 1. Graphs of loading level functions at values of parameter of non-linearity: $m = 0$ (1), 0·2 (2), 0·5 (3), 2·0 (4), 5·0 (5)

Thus, the settlement of a non-linearly deformable base surface under the external vertical load $p(x, y)$, distributed along the finite domain S, may be presented by the continuous summation in the following generalized form:

$$W(x, y) = \frac{1 - \nu^2}{\pi E} \int\int_S p(\xi, \eta)\,\bar{\varphi}(\xi, \eta, p_*)\,\omega(x, y, \xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{3}$$

where $\bar{\varphi}(\xi, \eta, p_*) = \varphi(p, p_*)/p$. Representation (3) permits the development of boundary contact problems for non-linearly deformable bases at the given dependence of $\varphi(p, p_*)$ or $\bar{\varphi}(p, p_*)$ and various kernels of the fundamental solutions $\omega(x, y, \xi, \eta)$ well documented in literature.

It should be noted that in Reference 9 account has been taken of the non-linear dependence of base settlement on loads in estimating rigid beams on heterogeneous Winkler's base obtained from the approximate formula

$$W = \frac{1 - \nu^2}{E} \omega \cdot b \frac{p}{1 - p/p_*} \tag{4}$$

where ω is a coefficient depending on the relation of sides of a rectangular foundation, and p_* is the pressure corresponding to the loss of base bearing capacity. Formula (4) was also used for determining coefficients of increasing normative pressures on soil for underground structures constructed in mountain regions. In addition, it was used for estimating the influence of non-linear relation between the settlements of foundations and loads acting on them in determining stresses in building structures on territories of underground works. In Reference 10 the 'hit and miss' method is used for solving the problem of calculating the rectangular plate resting on non-linearly deformable base being simulated by a continous half-space with a deformation modulus which is the function of the load

$$E = E_0 \left[1 - \left(\frac{p}{p_*}\right)^{\gamma}\right]^{1/\gamma}$$

where E_0 is an initial deformation modulus corresponding to the proportional dependence between the pressure on soil and displacements at its surface, p_* is a limit pressure on soil being determined by punch test; and γ is a non-linearity parameter.

NON-LINEAR INTEGRAL EQUATION SYSTEM OF CONTACT PROBLEM FOR ABSOLUTELY RIGID PUNCHES OF COMPLEX SHAPE WITH FLAT BASE

We will apply spatial contact model (3) together with the functions $\varphi(p, p_*)$ and $\omega(x, y, \xi, \eta)$ for determining contact pressures $p(x, y)$, vertical displacements W_c and slopes ψ_x, ψ_y of rigid punches of arbitrary forms situated on the non-linearly deformable base surface. We consider the punch to experience the action of a static load brought to the vertical resulting force **P** and the moments $\mathbf{M}_x$ and $\mathbf{M}_y$. We also assume the vertical punch displacements and base surface displacements to be equal, and tangential stresses at the contact surface to be absent.

For the spatial contact problem the equality of the vertical punch displacements of an area F and base surface leads to the following non-linear boundary-integral equation for determining unknown contact pressure and parameters of punch displacements as rigid body:

$$\iint_F p(\xi, \eta)\,\bar{\varphi}(p, p_*)\,\omega(x, y, \xi, \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = W_c + \psi_y \cdot (x - x_c) + \psi_x \cdot (y - y_c) \tag{5}$$

where F is the contact area between the punch and the base, $p(x, y)$ is the sought-for function of contact pressures, W_c is a vertical displacement of the punch centre, x_c, y_c are co-ordinates of the reduction centre of external forces. On a free base surface ($z = 0$) outside the domain of loading $p(x, y) = 0$. Furthermore, the system of equilibrium equations must be satisfied for punch:

$$\iint_F p(\xi, \eta)\xi\,\mathrm{d}\xi\,\mathrm{d}\eta = \mathbf{P}$$

$$\iint_F p(\xi, \eta)\xi\,\mathrm{d}\xi\,\mathrm{d}\eta = \mathbf{P}\cdot x_c - \mathbf{M}_y, \quad \iint_F p(\xi, \eta)\,\eta\,\mathrm{d}\xi\,\mathrm{d}\eta = \mathbf{P}\cdot y_c + \mathbf{M}_x \tag{6}$$

The exact solutions of the formulated spatial contact problem are available only for homogeneous isotropic linearly deformable half-space, when the contact domain has the shape of an ellipse or a circle in the simplest case of a centrally applied vertical force,[3,4,11,12] and also in the case of vertical load applied with eccentricity to a circular in plan, rigid punch.[13] For punches having a more complex shape and interacting with non-linearly deformable base we will use a numerical method which is successfully employed in the case of linearly deformable bases.[14–16]

Non-linear integral equation (5) is solved numerically simultaneously with conditions (6). For this purpose we divide the contact domain into N triangular and quadrangular elements. In the simplest case we assume a piecewise constant approximation of the function of contact pressures. As a result, in the limits of a separate element, $p(x, y) = const.$ If it is impossible to subject the contact domain to discretization into a sufficiently large number of elements, the calculations should be done with the use of a piecewise linear function of contact pressures. It is also done to increase the accuracy of the numerical solution.

In equation (5) we sequentially substitute co-ordinates of gravity centres of all elements, and double integrals on domain F are replaced by the sum of integrals in each element. Unknown contact pressures p_i on elements ($i = 1, \ldots, N$) as well as parameters W_c, ψ_x and ψ_y of punch displacement as a rigid body are defined by the system of ($N + 3$) non-linear (p_i variables)

algebraic equations:

$$
\begin{aligned}
&p_1 \bar{\varphi}(p_1, p_*)\delta_{i1} + p_2 \bar{\varphi}(p_2, p_*)\delta_{i2} + \cdots + p_N \bar{\varphi}(p_N, p_*)\delta_{iN} - W_c \\
&\quad - \psi_y \cdot (x_i - x_c) - \psi_x \cdot (y_i - y_c) = 0, \quad i = \overline{1, N} \\
&p_1 \Delta s_1 + p_2 \Delta s_2 + \cdots + p_N \Delta s_N = \mathbf{P} \\
&p_1 \Delta s_1 x_1 + p_2 \Delta s_2 x_2 + \cdots + p_N \Delta s_N x_N = \mathbf{P} \cdot x_c - \mathbf{M}_y \\
&p_1 \Delta s_1 y_1 + p_2 \Delta s_2 y_2 + \cdots + p_N \Delta s_N y_N = \mathbf{P} \cdot y_c + \mathbf{M}_x
\end{aligned} \tag{7}
$$

Herein $\delta_{ij} = \iint_{F_j} \omega(x_i, y_i, \xi, \eta)\, d\xi\, d\eta$ is vertical displacement of the surface of an elastic linearly deformable base at a point (x_i, y_i) coincident with the gravity centre of an ith element from a single load uniformly distributed over the domain F_j of jth element, Δs_i is the area of an ith element.

Numerical computations of the formulated problem were performed on PC IBM/AT-286 using software made up on FORTRAN-77 for punches of arbitrary shape in plan. In accordance with the developed algorithm the contact domain is sequentially subjected to discretization, coefficients δ_{ij} are calculated at a given function $\omega(x, y, \xi, \eta)$ determining the distributive base capacity, the equation system (7) is formulated and solved for different values of external force factors.

Calculation of coefficients δ_{ij} presents a certain difficulty because it is performed by a semi-analytical method. Contact elements are previously divided into triangles with common corners at points of singularities which present gravity centres of individual elements. For arbitrary triangular domain the summand in the kernel $\omega(x, y, \xi, \eta)$ corresponding to Boussinesq solution is integrated analytically.[17] The intergration of the remaining summands as well as coefficients δ_{ij}, $i \neq j$ is performed numerically by cubature formulae of the highest degree of accuracy of various orders.[18]

4. ITERATIVE PROCESSES OF SOLVING FINITE DIMENSIONAL ANALOG OF SPATIAL CONTACT PROBLEM FOR NON-LINEARLY DEFORMABLE BASE

As a result of contact domain discretization and piecewise constant approximation of the field of contact pressures, we have a finite dimensional analogue of a spatial contact problem in the form of N non-linear equations system (7) obtained from non-linear integral equation (5). We have, in addition, three linear equilibrium equations (6).

When $m = 0$, i.e. when the contact model is linear and the settlement of any point of the base surface is directly proportional to contact pressures, equations (7) become the system of linear algebraic equations and because of the diagonal predominance they have good conditionality. This fact, in its turn, allows us to use standard methods of the Gaussian type for solution without resorting to special regularization procedures. In this case it became possible to develop a rather common approach for solving spatial problems of contact interaction taking account of unilateral constraints for punches of a complex shape in plan on non-classical linearly deformable bases.[14-16].

When $m \neq 0$, system (7) is non-linear. It may be solved in a variety of ways but we chose the iterative approach of References 19–21. On the basis of experience of developing various iterative algorithms we can conclude that they are effective for solving the type of problems currently under consideration.

For system (7) the method of simple iterations is easier to solve. First, we solve system (7) for the case of linearly deformable base ($m = 0$). In this case all $\bar{\varphi}(p, p_*) = 1$. The obtained solution

$(p_1^0, p_2^0, \dots, p_N^0; W_c^0, \psi_x^0, \psi_y^0)$ is used as an initial approximation in the following iterative process:

$$\begin{aligned}
&p_1^{\alpha+1}\bar{\varphi}(p^\alpha, p_*)\delta_{i1} + p_2^{\alpha+1}\bar{\varphi}(p^\alpha, p_*)\delta_{i2} + \cdots + p_N^{\alpha+1}\bar{\varphi}(p^\alpha, p_*)\delta_{iN} - W_c^{\alpha+1} \\
&\quad - \psi_y^{\alpha+1}(x_i - x_c) - \psi_x^{\alpha+1}(y_i - y_c) = 0, \quad i = \overline{1, N} \\
&p_1^{\alpha+1}\,\Delta s_1 + p_2^{\alpha+1}\,\Delta s_2 + \cdots + p_N^{\alpha+1}\,\Delta s_N = \mathbf{P} \\
&p_1^{\alpha+1}\,\Delta s_1\, x_1 + p_2^{\alpha+1}\,\Delta s_2\, x_2 + \cdots + p_N^{\alpha+1}\,\Delta s_N\, x_N = \mathbf{P}\cdot x_c - \mathbf{M}_y \\
&p_1^{\alpha+1}\,\Delta s_1\, y_1 + p_2^{\alpha+1}\,\Delta s_2\, y_2 + \cdots + p_N^{\alpha+1}\,\Delta s_N\, y_N = \mathbf{P}\cdot y_c + \mathbf{M}_x
\end{aligned} \tag{8}$$

where α is an iteration number. Numerous computations showed that the iterative process (2), invariably converged for different parameter values of m and punches of various configurations in the conditions of a wide choice of well-known influence functions $\omega(x, y, \xi, \eta)$ and parameters of external loading. However, as it follows from the practice of numerous computations, the direct application of the simple iteration method is rather inefficient because of its slow convergence. Moreover, when the values of parameter m are increased the degree of convergence drops sharply. Therefore, for accelerating the convergence of approximations succession we used δ^2-Eitken transformation,[20,22] which is convenient for calculation and which permits a tenfold acceleration of the convergence of the iterative process (8) for the type of problems under consideration. The analysis carried out showed that the application of such iterative processes as Jacobi's and Seidel's non-linear relaxation methods[21] revealed no advantages compared to the method of simple iterations and had linear convergence. At the same time differential properties of non-linear functions entering the equations of system (7) allow us to use the Newton method and its modifications for solving this equation system with any preassigned degree of accuracy.

We will develop Newton method for system (7) of non-linear equations in the following manner. If p^α. $i = \overline{1, N}$; $W_c^\alpha, \psi_x^\alpha, \psi_y^\alpha$ are already known, then

$$\begin{aligned}
&p_i^{\alpha+1} = p_i^\alpha + \Delta p_i^\alpha, \quad i = 1, \dots, N \\
&W_c^{\alpha+1} = W_c^\alpha + \Delta W_c^\alpha, \quad \psi_x^{\alpha+1} = \psi_x^\alpha + \Delta\psi_x^\alpha, \quad \psi_y^{\alpha+1} = \psi_y^\alpha + \Delta\psi_y^{\alpha+1}
\end{aligned}$$

where the values Δp_i^α, $i = \overline{1, N}$; $\Delta W_c^\alpha, \Delta\psi_x^\alpha, \Delta\psi_y^\alpha$ are defined by the following system of linear equations:

$$\begin{aligned}
&\delta_{i1}\varphi'(p_1^\alpha)\Delta p_1^\alpha + \delta_{i2}\varphi'(p_2^\alpha)\Delta p_2^\alpha + \cdots + \delta_{iN}\varphi'(p_N^\alpha)\Delta p_N^\alpha \\
&\quad - \Delta W_c^\alpha - \Delta\psi_y^\alpha(x_i - x_c) - \Delta\psi_x^\alpha(y_i - y_c) = -\delta_{i1}\varphi(p_1^\alpha) \\
&\quad - \delta_{i2}\varphi(p_2^\alpha) - \cdots - \delta_{iN}\varphi(p_N^\alpha) + W_c^\alpha + \psi_y^\alpha(x_i - x_c) \\
&\quad + \psi_x^\alpha(y_i - y_c), \quad i = 1, 2, \dots, N \\
&\Delta p_1^\alpha\,\Delta s_1 + \Delta p_2^\alpha\,\Delta s_2 + \cdots + \Delta p_N^\alpha\,\Delta s_N = \mathbf{P} - p_1^\alpha\,\Delta s_1 - p_2^\alpha\,\Delta s_2 - \cdots - p_N^\alpha\,\Delta s_N \\
&\Delta p_1^\alpha\,\Delta s_1\, x_1 + \Delta p_2^\alpha\,\Delta s_2\, x_2 + \cdots + \Delta p_n^\alpha\,\Delta s_N\, x_N = \mathbf{P}\cdot y_c - \mathbf{M}_y - \Delta s_1\, x_1 p_1^\alpha \\
&\quad - \Delta s_2\, x_2\, p_2^\alpha - \cdots - \Delta s_N\, x_N\, p_N^\alpha \\
&\Delta p_1^\alpha\,\Delta s_1\, y_1 + \Delta p_2^\alpha\,\Delta s_2\, y_2 + \cdots + \Delta p_N^\alpha\,\Delta s_N\, y_N = \mathbf{P}\cdot x_c + \mathbf{M}_x - \Delta s_1\, y_1 p_1^\alpha \\
&\quad - \Delta s_2\, y_2\, p_2^\alpha - \cdots - \Delta s_N\, y_N\, p_N^\alpha
\end{aligned} \tag{9}$$

As an initial approximation we also use the solution corresponding to the case of a linearly deformable base. Due to the choice of such an initial approximation and the quadratic convergence of the method, the succession of approximations converges rather quickly (3–5 iterations) in

a wide range of modifications of non-linear parameter m. It should be noted that in case of values $m \geqslant 5$, the aforementioned processes of approximation give stable solutions together with the method of loading along the parameter m.[23]

It is apparent that numerical solutions p_i, $i = \overline{1, N}$, of integral equation system (7) have a clear physical meaning of contact stresses. As the function $\bar{\varphi}(\xi, \eta, p_*)$ is dimensionless, the dimensionality $p(\xi, \eta)$ is not changed with the use of a non-linear model. In addition, solutions of a non-linear contact problem, apart from integral representations (3), are certain to satisfy integral conditions of equilibrium (5). Finally, in limiting cases $p_* \to \infty$, or $m \to 0$ the solutions being obtained coincide with contact stresses found in a classical problem of the theory of elasticity.

Following from the theoretical investigation presented, the physical meaning of the function $\omega(x, y, \xi, \eta)$ in a non-linear contact model is not changed, i.e. it is assumed to be taken from the consideration of non-linear laws of deformation. It is necessary to stress that the semi-empirical model used in form (3) should be considered as a complex representation of interdependence between the load and the settlement in an integral meaning using a dimensionless function of loading level $\bar{\varphi}(p, p_*)$. The functions $p(\xi, \eta)$ and $\omega(x, y, \xi, \eta)$ have their initial meaning (from the linear theory) of a contact pressure and the influence function for linearly deformable base which is not subjected to any modifications.

According to the numerical algorithm being proposed, the computations are performed by a unified technique which is suitable both for the cases of linear and non-linear deformations. Solution of the influence function is done in a separate programmed modulus that does not require the modification of the algorithm as a whole.

5. CONTACT PROBLEM FOR CIRCULAR PUNCH ON NON-LINEARLY DEFORMABLE BASE

As an example we will consider a spatial contact problem for circular, in plan, punch interacting with a non-linearly deformable base. We will restrict our consideration to the punch loaded only by the vertical force **P** as is customary when punch tests are being conducted.

In order to numerically solve the problem we subject a circular contact domain to discretization (Figure 2) into triangular and quadrangular elements with radii and concentric circumferences being concentrated at the boundary, that is based on a sharp change of the stressed condition of the base near the punch edge. The radii of concentric circumferences are calculated by means of the following quasiuniform function:

$$r = a\frac{e^{\beta t} - 1}{e^{\beta} - 1}, \quad t_l = \frac{l - 1}{L}, \quad l = 1, \dots, L.$$

The degree of concentrations is regulated by the choice of the parameter β.

In the case when the influence function is symmetric, $\omega(x, y, \xi, \eta) = \omega(x - \xi, y - \eta)$, system (7) for a circular punch is substantially simplified by axial symmetry, at the expense of decreasing dimensionality, and it takes the form

$$\sum_{l=1}^{L} A_{il} p_l \bar{\varphi}(p_l, p_*) - W_c = 0, \quad i = \overline{1, L}$$

$$\sum_{l=1}^{L} p_l \, \Delta s_l = \frac{P \cdot L}{N} \tag{10}$$

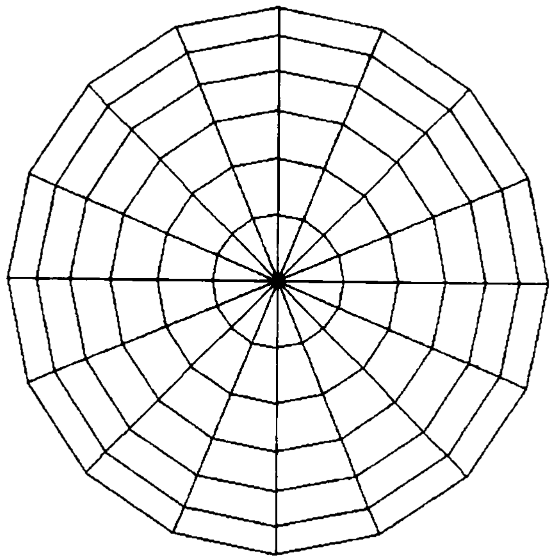

Figure 2. Discretization of circular contact area

where A_{il} is defined with the use of influence coefficients δ_{ij}, $i, j = 1, \dots, N$, by the formula

$$A_{il} = \sum_{K=1}^{N/L} \delta_{i,l+L(K-1)}$$

The dimensionality of equation system (10) is equal to $L+1$, where L is the number of contact elements in the radial direction.

The following sections consider the numerical simulation results of the contact interaction process of a circular punch for the most practical models of elastic bases of a half-space type and a finite thickness layer. The urgency of conducting such investigations is dictated by the necessity of restoring non-linear deformation diagrams according to the data of direct contact experiments.

5.1. Non-linearly deformable half-space

We will use the well-known Boussinesq solution concerning the action of a concentrated unit force normal to the surface of an elastic linearly deformable half-space.[3] Herein, the influence function takes the form

$$\omega(x, y, \xi, \eta) = \frac{1-\nu^2}{\pi E} \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} \tag{11}$$

In the case of the interaction of a centrally loaded punch with a linearly deformable half-space, the problem has the analytical solution[11]

$$W_0 = \frac{\mathbf{P}(1-\nu^2)}{2Ea}, \quad \psi_x = \psi_y = 0, \quad p(r) = \frac{\mathbf{P}}{2\pi a\sqrt{a^2 - r^2}}$$

where a is the punch radius, r is the distance from the centre to any point under the punch. This solution presented a test the algorithm developed. Sufficiently accurate values of contact stresses

in test calculations[14] were obtained at $\beta = -1{\cdot}0$ and the discretization of the circle domain performed with the use of 96 elements which were formed by six concentrated circumferences and 16 rays (Figure 2).

The computation results of contact pressures and displacements for a circular punch on a non-linearly deformable half-space are presented in a dimensionless form in Figures 3–5. As shown in Figure 3, with an increase in the parameter of non-linearity m, the graphs of contact pressures become flatter and the main change of contact pressures takes place near the punch edge. Relative settlements of the punch at a given load with an increase in parameter m are

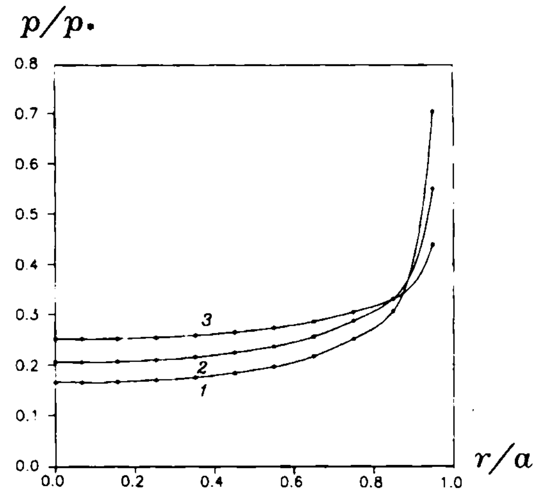

Figure 3. Graphs of contact pressures under the circular punch for non-linearly deformable half-space at $\mathbf{P} = a^2 p_*$, $m = 0$ (1), 0·8 (2), 3·0 (3)

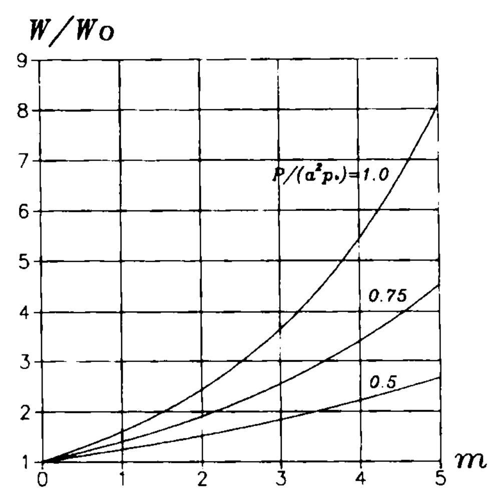

Figure 4. Dependence of relative settlements of a circular punch on a non-linearly deformable half-space on the parameter of non-linearity m at various loading levels

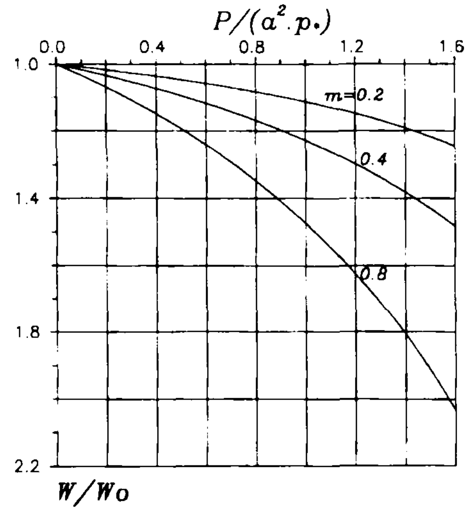

Figure 5. 'Load-settlement' graphs for a circular punch on a non-linearly deformable half-space on the parameter of non-linearity m

growing non-linearly, as shown in Figure 4. Figure 5 shows the non-linear 'load-settlement' dependencies at different values of the parameter m.

5.2. Non-linearly deformable finite thickness layer

In spite of its wide application, the model of linearly deformable half-space has, nevertheless, some disadvantages. In particular, the assumption that soil base is a spatially uniform semi-infinite medium appears to be especially weak. Such an assumption essentially idealizes the situation and it is not always confirmed by practical computations. Using the model of linearly deformable finite thickness layer correlates, to a considerably large extent, with the data on foundation settlements on soil base and enables to reduce design values of stresses and deformations in structures situated on an elastic base.

In accordance with[12], the influence function $\omega(x, y, \xi, \eta)$, defining the vertical displacements of the surface points of the linearly deformable finite thickness layer caused by the action of a single normal concentrated force on its surface, is determined by the following integral dependence:

$$\omega(x, y, \xi, \eta) = \frac{1 - v^2}{\pi E} \int_0^\infty Q(H, t) \cdot J_0(R \cdot t)\, \mathrm{d}t \tag{12}$$

where $R = \sqrt{(x - \xi)^2 + (y - \eta)^2}$, J_0 is zero-order Bessel function of the first kind.

The kernel $Q(t)$ of the integral representation (12) has the form

$$Q(H, t) = \frac{2sh^2(Ht)}{2Ht + sh(2ht)} \tag{13}$$

where $0 < H < \infty$ is the thickness of an unlimited elastic layer situated on an absolutely rigid base. Expression (13) is obtained using Hankel's transformation on the assumption that no friction forces exist between the contact of the layer and the base. For the uniform elastic half-space, $Q = 1$.

The model of an elastic layer of a finite thickness is, in a mathematical sense, more common than that of an elastic half-space. At sufficiently large thicknesses of the layer $H > a$, representation (12) transfers into (11), i.e. a computations based on formula (13), lead to the solution of contact problems for half-space.

It is widely known that,[24] in the case of the interaction of centrally loaded punch with a linearly deformable finite thickness layer, the solution of contact problem is based on its reduction to the double integral equations. It permits us to express the punch displacements and contact stresses by an auxiliary function satisfying Fredholm's integral equation with the continuous symmetric kernel. Although it is possible to obtain the solution of this equation with a necessary degree of accuracy, it is done only numerically.

In solving the problem of a contact interaction of punches, having an arbitrary shape in plan, with an elastic finite thickness layer under the complicated spatial loading, the use of the integral representation (12) according to the proposed numerical algorithm is connected with additional calculating difficulties in the computation of improper integrals continuing Bessel oscillating functions. Therefore, while solving contact problems for a finite thickness layer, it is convenient to present integrand (13) by means of a finite series of exponential functions. It allows us to calculate improper integrals analytical and thereby to raise the accuracy of calculations and at the same time to make considerable time savings. In particular, the highly accurate approximation

$$\frac{2\cdot sh^2(\alpha)}{2\alpha + sh(2\alpha)} = 1 + \sum_{k=1}^{4} B_k \exp(-A_k\cdot\alpha)$$

carried out by the method of least squares in Reference 25 permits us to use approximate solution[5] for a concentrated normal load on a layer surface of H thickness in the form

$$\omega(x, y, \xi, \eta) = \frac{1-\nu^2}{\pi E}\left(\frac{1}{R} + \sum_{k=1}^{4} \frac{B_k}{\sqrt{(A_k H)^2 + R^2}}\right) \tag{14}$$

where coefficients A_k, B_k, $k = \overline{1, 4}$ are given in the Table I.

The first term in equation (14) corresponds to the Boussinesqu's solution, and the remaining summands which depend on the layer thickness H may be considered as correcting terms. It is important to note that the function $Q(H, t)$ being used is defined by one parameter of a length dimensionality (by conditionally compressed thickness H) which is a unique geometric base parameter. As a consequence the coefficients in (14) are fixed and, what is more important, do not depend on the characteristics of mechanical properties of the base and its depth.

The conducted numerical investigations of contact interaction of a circular punch for a non-linearly deformable finite thickness layer are qualitatively coordinated with the design data for a half-space and are partially mapped in Figures 6–8. As we might expect, the graphs of contact pressures become more plane both with increasing the parameter of non-linearity m and with decreasing the thickness of the compressible layer H (or the parameter $s = H/a$). Relative punch settlements at a given load with increasing parameters m and s grow non-linearly without

Table I. Approximating parameters of a contact model for an elastic constant thickness layer

k	1	2	3	4
A_k	0, 8	1, 4	2, 0	2, 6
B_k	0, 426	− 6, 051	7, 395	− 2,770

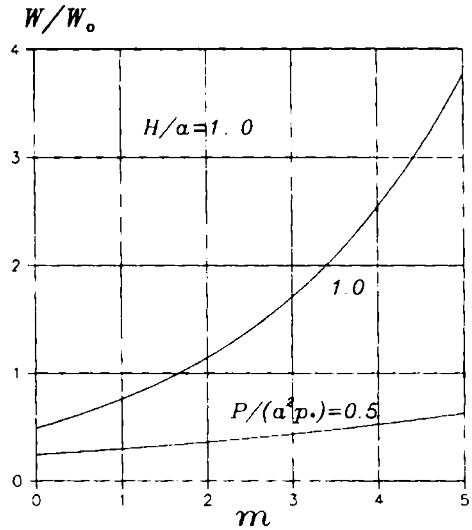

Figure 6. Dependence of relative settlements of a circular punch on a non-linearly deformable finite thickness layer on the parameter of non-linearity m at various loading levels

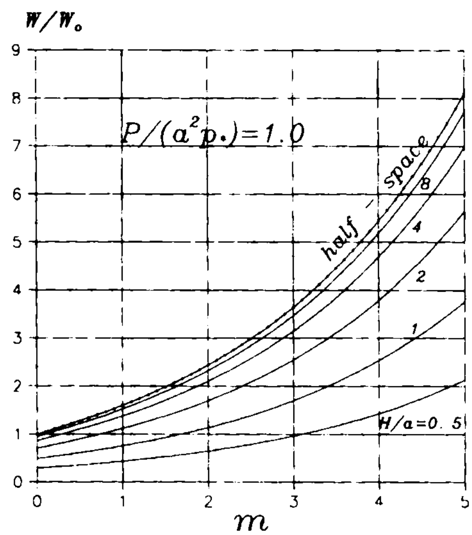

Figure 7. Dependence of relative settlements of a circular punch on a non-linearly deformable finite thickness layer on the parameter of non-linearity m at various depth of a compressible thickness, $\mathbf{P} = a^2p_*$

exceeding appropriate values for the half-space (Figure 6). With increasing the load the noted effect is evident to a greater extent (Figure 7). The non-linear character of graphs 'load-settlement' at different values of parameters m and H is illustrated in Figure 8.

As already noted above, the conducted computations were obtained on the assumption that there is no friction between the elastic layer and the rigid base. However, the presence of contact surface between the elastic layer and the incompressible base in soil at the depth H leads to heterogeneity of the stress–strain state. A number of authors [26] have found that the influence of

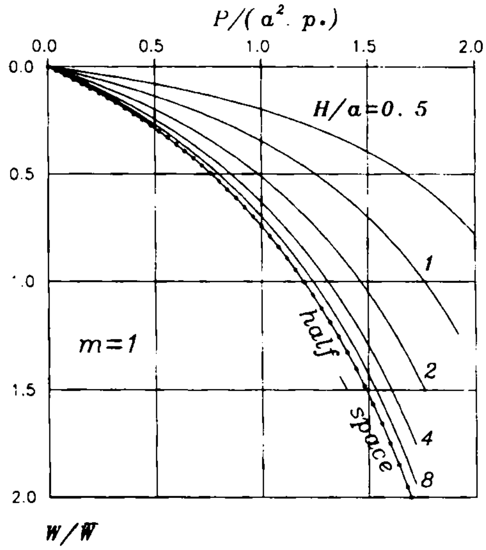

Figure 8. 'Load-settlement' graphs for a circular punch on a non-linearly deformable finite thickness layer, $\bar{W} = (1 - \nu^2)ap_*/2E$

an incompressible base on the concentration of stresses inside the layer becomes insignificant only at $s = H/a > 5$. The degree of concentration at $s < 5$ essentially depends on the conditions of the elastic layer slide along the incompressible base. In particular, detailed analysis shows that the absence of tangential stresses, as compared with that of the absence of displacements (restraint) on the lower boundary of the layer, causes an increase in vertical displacements and compressible stresses. Obviously, for real soil conditions, there is a need to use the boundary condition[27] of the type of elastic constraints between the elastic and absolutely rigid bodies at the contact

$$u_j(x, y, H) = \frac{1 - \nu}{G} \mu\tau_{jz}(x, y, H), \quad j = x, y, \quad u_Z(x, y, H) = 0$$

where μ is an adhesion coefficient leading to the growth of absolute values of stresses and displacements. The limiting cases of such conditions are the adhesion and smooth contact: at $\mu = 0$ there is the adhesion of the layer and the base; as $\mu \to \infty$ the layer and the base can slip without friction along all their common boundary

As a further application to real conditions of deformation, the simulation of a natural base in the form of the scheme 'an elastic finite thickness layer on an elastic half-space', should be recognized. For example, when an elastic layer freely ($\mu = 0$) lies on elastic half-space, the influence function has the form[28]

$$Q(\alpha, \chi) = \frac{ch2\alpha - 1 + \chi(2\alpha + sh2\alpha)}{2\alpha + sh2\alpha + \chi(ch2\alpha - 1 - \alpha^2)} \tag{15}$$

This base model is characterized by the relation $\chi = \theta_1/\theta_2$, where $\theta_i = E_i/2(1 - \nu_i^2)$, $i = 1, 2$, are mechanical characteristics of the layer and the half-space, respectively. At $\chi \to 0$ we arrive at the scheme 'an elastic layer on a rigid base', at $H/a \to \infty$ — at the scheme of a uniform half-space.

According to the afore-mentioned modulus structure of the numerical algorithm developed, the use of the influence function for the generalized model of linearly deformable base (15) will not lead to increase in fundamental calculation difficulties, but it will manifest itself only in the general increase in computation time.

6. ESTIMATION OF NONLINEAR DEFORMATION EFFECTS ACCORDING TO THE RESULTS OF PUNCH TESTS

From the afore mentioned computation data it follows that the contact model being considered can be employed upon identification of parameters m, p_* and H using a database of punch tests, for the description of deformation characteristics of bases. It can, in addition, be used for estimating spatial contact interaction of foundation structures with the soil on the non-linear stage of deformation.

The qualitative character of the calculated dependencies (Figures 5 and 8) points to the suitability of the model for the description of non-linear deformations under the foundations of clay soils of soft or hard plastic consistency as well as sandy soils of an average density with the deformation diagrams without phenomena of hardening.

One of the possible algorithms for treating punch test data as applied to the considered model of non-linearly deformable base is presented below.

First, we will estimate the value of parameter p_* according to the results of conventional punch tests,[29] or we will use for the first approximation the well-known formulae of Beresantsev[30] or of Egorov[31] for the initial critical load onto the soil in the case of circular punches.

Next, the computation results given in Figures 4, 6 and 7 represent particular diagrams of loading for the series of loads $\mathbf{P}_i = K_i \cdot (a^2 p_*)$ ($i = 1, 2, \ldots, M$, M is the number of experimental points on the graph 'load-settlement'). If we conduct experiments for these $\mathbf{P}_i$, then according to the available experimental values $(W)_i$, it is easy to define appropriate values $\{m_i\}$ using design diagrams. Afterwards, a standard statistical treatment[32] of the array of obtained values $\{m_i\}$ and the estimation of a probable mean value m for it were carried out. If the scatter in the experimental data is small, and if the correlation of the calculated values to the empirical data is performed according to Fisher's criterion at a given significance level, then the model is adequate to the experimental data on the basis of corresponding goodness-of-fit test, and the value $\bar{m}$ is taken as a non-linear parameter of the model.

The proposed theoretical analysis of spatial contact interaction on non-linear deformation stage suggests that p_* and m are not independent parameters of a contact model, as the choice of the formula for the limiting contact pressure p_* or its experimental definition will affect the estimation of $\bar{m}$.

The results of our analysis (Figure 8) agree with well-known practical observational data that one of the factors which essentially affects the settlement value is the depth H of the compressible thickness of the soil. The depth is defined rather accurately when concrete engineering and geological conditions are known. The most widespread examples occur in situations where, for example, practically incompressible rocks exist at a certain depth under the compressible soil. Every so often, under construction, seasonal thawing occurs to a finite depth, below which permafrost soils are situated. In this case the entire compressible layer is naturally considered to be of compressible thickness. Moreover, there is a large number of calculation methods[26] according to which the depth of a compressible thickness is defined: by foundation width, by the punch tests together with the solution of the theory of elasticity, by comparison of natural and excessive pressures, etc. The most widespread method for defining the value of thickness H is its

conditional evaluation as the depth where an excessive punch pressure (of the foundation model) constitutes 10–20% of that which would occur naturally at the same depth. However, none of the well-known methods of defining the depth of the compressible thickness is free from defects. The depth values of the compressible thickness, found for identical conditions by different methods, sometimes diverge by 2 or 5 times. Thus, the compressible thickness is a conditional value, which is introduced into the calculation because of the difference of real conditions from the calculated model, and it must be assigned on the basis of the coincidence requirement of the design and actual settlements. Therefore, the value H is conveniently included into a number of non-linear parameters of the contact model and its value can be derived from the approximation conditions of design settlements and natural ones. Using such an approach the non-linear character of the observable growth of the thickness with increasing load, the influence of the punch area, soil compactness and some other factors can be naturally taken into account.

Non-linear parameters of m, n, H model can, in aggregate, be found by minimizing the squared error in the mathematical treatment of the punch test results by one of the methods of non-linear programming.[33] One example of this approach is the simplex method when a direct search using only function values is carried out.

It is necessary to point out the relation of parameters p_* and m with the function $\omega(x, y, \xi, \eta)$. In accordance with the theory of linearly deformable base,[34] the influence function $\omega(x, y, \xi, \eta)$ defines distributive properties of soil bases for the calculated model which in practice replaces the real natural base. Nowadays a number of these functions is widely used in designing and calculating structures on an elastic base, and they are included into the acting normative documents.

Linearly deformable half-space, linearly deformable finite thickness layer, and Winkler's model are widely used in design practice. The influence functions for many elastic non-classical bases in considering, for example, layered rock stratification,[7] the change of the elastic properties depending on the depth,[8] etc. have been obtained. It is apparent that at a given punch form the choice of $\omega(x, y, \xi, \eta)$ defines a characteristic diagram 'load-settlement' at different values of parameters p_* and m and thereby evaluate the parameters of non-linearity as a result of experimental data treatment. In this paper computations are conducted for more commonly used contact models with influence functions (11) and (12). For circular punches the choice of a new function $\omega(x, y, \xi, \eta)$ will lead to the other design diagrams of deformation, on the basis of which model parameters p_*, m and n will be identified after the treatment of experimental data of the punch tests. The fact that the function $\omega(x, y, \xi, \eta)$ has additional free parameters (for example, parameter H — the depth of the conditionally compressible thickness, or γ — the degree of heterogeneity of mechanical properties in depth, etc.), permits to raise the approximation accuracy of the experimental data. The example of non-linear contact problem for an elastic layer of a finite thickness, discussed above in Section 4.2, shows clearly and convincingly a strong connection of the parameters p_*, m with the function $\omega(x, y, \xi, \eta)$ (especially computations given in Figures 6–8).

For complete consideration of the problem, mention must be made of the choice of influence functions $\omega(x, y, \xi, \eta)$ themselves. It is a well-known fact[11] that influence function $\omega(x, y, \xi, \eta)$ which would enable us to describe the behaviour of different soils in a wide range of deformation conditions has not been found. Data presented in the literature, on the results of the comparison of various influence functions, do not allow us to determine which of the available influence functions is best for satisfactory description of displacements and stresses for structures situated on elastic bases. Though each influence function has its own physical justification, at present it is impossible to prefer any of these functions. Therefore, the choice of influence function $\omega(x, y, \xi, \eta)$ is limited by personal intuition of an investigator on the basis of a reasonable compromise

between the complexity of mathematical representations and the data of predicting the performance of foundations under buildings and structures. We will note once again that the application of one or another influence function $\omega(x, y, \xi, \eta)$ essentially affects the solution of the contact problem (Figures 6–8) or the degree of non-uniform distribution of reactive pressures which influence the change of loading diagrams $W(m, H, p_*)$. The latter ones are used for estimating the parameters of non-linearity.

It is important to note that free parameters of non-linearity m, H and p_* of a semiempirical model are always found from the condition of the best description of experimental data on the non-linear deformation stage. Such approach is generally recognized and widely used for analysis of non-linear mechanical models, for example, in semiempirical theory of turbulence, and in rheology of polymeric fluids and solids. Free parameters of mechanical models characterizing non-linear properties of material bear the special name of parameters of non-linearity, indeterminate coefficients, fitting parameters, etc. Such approach is more convincingly demonstrated in a number of papers[35–37] which are dedicated to the elaboration of the design of non-linear mechanical models. It is not difficult to cite a number of other papers where the usefulness of the semiempirical approach for the description of various non-linear mechanical phenomena is clearly demonstrated. It is necessary to point out two more notable papers which deal with calculating structures on an elastic base.[9,10] In these papers, the concept used by the authors was earlier applied for calculating structures of the type of beams or plates on an elastic non-linearly deformable base.

7. CONCLUSIONS

The variety of soil properties and the complexity of the processes of their deformation reasonably justify the elaboration of computation methods using various, more precise, design models of contact interaction. The analysed contact model of the soil base is semiempirical. It connects, in a natural way, the parameters of non-linearity p_*, m, n and the base distributive properties prescribed by the function $\omega(x, y, \xi, \eta)$. In limiting case ($p_* \to \infty$, or $m \to 0$) the contact model is linear. It predicts a rectilinear diagram 'load-settlement', and it is entirely characterized by the chosen influence function $\omega(x, y, \xi, \eta)$. Thus, the physically non-linear contact model of integral type has been considered. It represents the generalization of linear spatial contact models for the conditions of a non-linear deformation. The design formula of model (3) includes the influence function for a linearly deformable base as its component. It should be noted that in formulating the proposed contact model note is not taken of the relation between stresses and deformations. Therein become apparent the characteristics and certain merits of the non-linear contact model of integral type (3), connecting displacements and contact stresses only on the surface of an elastic base. In this case the dimensionality of the problem is reduced, and it becomes two-dimensional in spatial statement. Equation (3) does not conjecture one-to-one correspondence with any rheological law unlike classical integral representations of the theory of elasticity. The inverse problem noted earlier as yet belongs to unsolved mechanics problems of continuous media and it has not been proposed in this paper. Here, the situation is quite analogous to the application of design contact models with one or two reaction moduli (Winkler's, Pasternak's or Philonenko–Borodich's) which also do not conjecture a corresponding dependence between the components of stress tensors and deformations. However, the practice of engineering calculations, non-linear contact models with dependence of reaction modulus on contact pressures or displacements, can be successfully used.[38]

NOTATION

a	radius of circular area
E	Young's modulus
F	contact area
G	shear modulus
H	layer depth
J_0	zero-order Bessel function of first kind
N	number of elements
$\mathbf{M}_x, \mathbf{M}_y$	resulting moments
$\mathbf{P}$	vertical load
$Q(t)$	kernel of integral representation of Green's function
S	contact area
W	settlement in Cartesian co-ordinate system
$m, n > 0$	dimensionless parameters of base
p	contact pressure
p_*	limit contact stress
x, y	Cartesian co-ordinate system
ξ, η	local co-ordinate system
μ	adhesion coefficient
ν	Poisson's ratio
$\varphi(p, p_*)$	loading level function
ψ_x, ψ_y	slopes of rigid punch
$\omega(x, y, \xi, \eta)$	displacement Green's function

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